

Uniform boundedness of pretangent spaces, local constancy of metric derivatives and strong right upper porosity at a point

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Abstract. Let (X, d, p) be a pointed metric space. A pretangent space to X at p is a metric space consisting of some equivalence classes of convergent to p sequences $(x_n), x_n \in X$, whose degree of convergence is comparable with a given scaling sequence $(r_n), r_n \downarrow 0$. A scaling sequence (r_n) is normal if this sequence is eventually decreasing and there is (x_n) such that $|d(x_n, p) - r_n| = o(r_n)$ for $n \rightarrow \infty$. Let $\Omega_p^X(\mathbf{n})$ be the set of pretangent spaces to X at p with normal scaling sequences. We prove that $\Omega_p^X(\mathbf{n})$ is uniformly bounded if and only if $\{d(x, p) : x \in X\}$ is a so-called completely strongly porous set. It is also proved that the uniform boundedness of $\Omega_p^X(\mathbf{n})$ is an equivalent of the constancy of metric derivatives of all metrically differentiable mappings on X in the open balls of a fixed radius centered at the marked points of pretangent spaces.

Keywords. pretangent space to metric space, boundedness, local strong one-side porosity, metric space valued derivative.

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This paper is dedicated to Prof. Matti Vuorinen on the occasion of his 65th birthday.

The research interests of Prof. Matti Vuorinen are highly wide. In particular, it includes both the theory of metric spaces and the infinitesimal geometry. The present paper lies in the intersection of these theories.

1. Introduction

Recent achievements in the metric space theory are closely related to some generalizations of the differentiation. A possible but not the only one initial point

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to develop the theory of a differentiation in metric spaces is the fact that every separable metric space admits an isometric embedding into the dual space of a separable Banach space. It provides a linear structure, and so a differentiation. This approach leads to a rather complete theory of rectifiable sets and currents on metric spaces [AK1, AK2]. The concept of the upper gradient [H, HK, SH], Cheeger's notion of differentiability for Rademacher's theorem in certain metric measure spaces [CH], the metric derivative in the studies of metric space valued functions of bounded variation [A1, A2] and the Lipschitz type approach in [HA] are important examples of such generalizations. The generalizations of the differentiability mentioned above give usually nontrivial results only for the assumption that metric spaces have "sufficiently many" rectifiable curves.

A new sequential approach to the notion of differentiability for the mapping between the general metric spaces was produced in [DM1] (see also [DM2]). The main idea of this approach is to introduce a "metric derivative" for the mapping between metric spaces by an intrinsic way which does not depend on the possible linear structure in these spaces even if such structures is given. A basic technical tool in [DM1] is pretangent and tangent spaces to an arbitrary metric space X at a point p . The development of the theory of differentiation in metric spaces without linear structure requires the understanding of interrelations between the infinitesimal properties of initial metric space and geometry of pretangent spaces to this initial.

It is almost clear that the boundedness of pretangent spaces to a metric space (X, d) at a point $p \in X$ is closely related to the porosity of the set $S_p(X) = \{d(x, p) : x \in X\}$ at the point zero. Another not so obvious hypothesis is that the uniform boundedness of all pretangent spaces (to X at p) leads to the "constancy of metric derivatives" of all "metrically differentiable" mappings on X near distinguished points of these pretangent spaces.

What type of porosity of $S_p(X)$ at 0 corresponds to the occasion?

The main result of the paper, Theorem 3.11, gives a complete answer to this question. Furthermore, in Proposition 5.4, after a formal definition of "metric derivatives", we also obtain a positive answer to the hypothesis formulated above.

Recall some results and terminology related to the pretangent spaces.

Let (X, d, p) be a pointed metric space with a metric d and a marked point p . Fix a sequence \tilde{r} of positive real numbers r_n tending to zero. In what follows \tilde{r} will be called a *scaling sequence*. Let us denote by \tilde{X} the set of all sequences of points from X .

Definition 1.1. Two sequences $\tilde{x} = (x_n)_{n \in \mathbb{N}}$ and $\tilde{y} = (y_n)_{n \in \mathbb{N}}$, $\tilde{x}, \tilde{y} \in \tilde{X}$ are mutually stable with respect to $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ if the finite limit

$$(1.1) \quad \lim_{n \rightarrow \infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y})$$

exists.

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. \tilde{r}) if any two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if \tilde{F} is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that \tilde{x} and \tilde{z} are not mutually stable.

A standard application of Zorn's lemma leads to the following

Proposition 1.2. *Let (X, d, p) be a pointed metric space. Then for every scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ there exists a maximal self-stable family $\tilde{X}_{p, \tilde{r}}$ such that $\tilde{p} := (p, p, \dots) \in \tilde{X}_{p, \tilde{r}}$.*

Note that the condition $\tilde{p} \in \tilde{X}_{p, \tilde{r}}$ implies the equality $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ for every $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}_{p, \tilde{r}}$.

Consider a function $\tilde{d} : \tilde{X}_{p, \tilde{r}} \times \tilde{X}_{p, \tilde{r}} \rightarrow \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, \tilde{d} is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_{p, \tilde{r}}$. Hence $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$ is a pseudometric space.

Definition 1.3. A pretangent space to the space X (at the point p w.r.t. \tilde{r}) is the metric identification of a pseudometric space $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$.

Since the notion of pretangent space is basic for the paper, we remind this metric identification construction.

Define a relation \sim on $\tilde{X}_{p, \tilde{r}}$ as $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then \sim is an equivalence relation. Let us denote by $\Omega_{p, \tilde{r}}^X$ the set of equivalence classes in $\tilde{X}_{p, \tilde{r}}$ under the equivalence relation \sim . It follows from general properties of the pseudometric spaces (see, for example, [K]) that if ρ is defined on $\Omega_{p, \tilde{r}}^X$ as

$$(1.2) \quad \rho(\gamma, \beta) := \tilde{d}(\tilde{x}, \tilde{y})$$

for $\tilde{x} \in \gamma$ and $\tilde{y} \in \beta$, then ρ is a well-defined metric on $\Omega_{p, \tilde{r}}^X$. By definition, the metric identification of $(\tilde{X}_{p, \tilde{r}}, \tilde{d})$ is the metric space $(\Omega_{p, \tilde{r}}^X, \rho)$.

It should be observed that $\Omega_{p,\tilde{r}}^X \neq \emptyset$ because the constant sequence \tilde{p} belongs to $\tilde{X}_{p,\tilde{r}}$. Thus every pretangent space $\Omega_{p,\tilde{r}}^X$ is a pointed metric space with the natural distinguished point $\alpha = \pi(\tilde{p})$, (see diagram (1.3) below).

Let $(n_k)_{k \in \mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by \tilde{r}' the subsequence $(r_{n_k})_{k \in \mathbb{N}}$ of the scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ and let $\tilde{x}' := (x_{n_k})_{k \in \mathbb{N}}$ for every $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}$. It is clear that if \tilde{x} and \tilde{y} are mutually stable w.r.t. \tilde{r} , then \tilde{x}' and \tilde{y}' are mutually stable w.r.t. \tilde{r}' and $\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}')$. If $\tilde{X}_{p,\tilde{r}}$ is a maximal self-stable (w.r.t \tilde{r}) family, then, by Zorn's Lemma, there exists a maximal self-stable (w.r.t \tilde{r}') family $\tilde{X}_{p,\tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{p,\tilde{r}}\} \subseteq \tilde{X}_{p,\tilde{r}'}$$

Denote by $in_{\tilde{r}'}$ the map from $\tilde{X}_{p,\tilde{r}}$ to $\tilde{X}_{p,\tilde{r}'}$ with $in_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{p,\tilde{r}}$. It follows from (1.2) that after metric identifications $in_{\tilde{r}'}$ passes to an isometric embedding $em' : \Omega_{p,\tilde{r}}^X \rightarrow \Omega_{p,\tilde{r}'}^X$ under which the diagram

$$(1.3) \quad \begin{array}{ccc} \tilde{X}_{p,\tilde{r}} & \xrightarrow{in_{\tilde{r}'}} & \tilde{X}_{p,\tilde{r}'} \\ \pi \downarrow & & \downarrow \pi' \\ \Omega_{p,\tilde{r}}^X & \xrightarrow{em'} & \Omega_{p,\tilde{r}'}^X \end{array}$$

is commutative. Here π and π' are the natural projections,

$$\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\} \quad \text{and} \quad \pi'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0\}.$$

Let X and Y be metric spaces. Recall that a map $f : X \rightarrow Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 1.4. A pretangent $\Omega_{p,\tilde{r}}^X$ is tangent if $em' : \Omega_{p,\tilde{r}}^X \rightarrow \Omega_{p,\tilde{r}'}^X$ is an isometry for every \tilde{r}' .

The following lemmas will be used in sections 3 and 4 of the paper.

Lemma 1.5. [DAK] *Let \mathfrak{B} be a countable subfamily of \tilde{X} and let $\tilde{\rho} = (\rho_n)_{n \in \mathbb{N}}$ be a scaling sequence. Suppose that the inequality*

$$\limsup_{n \rightarrow \infty} \frac{d(b_n, p)}{\rho_n} < \infty$$

holds for every $\tilde{b} = (b_n)_{n \in \mathbb{N}} \in \mathfrak{B}$. Then there is an infinite subsequence $\tilde{\rho}'$ of $\tilde{\rho}$ such that the family $\mathfrak{B}' = \{\tilde{b}' : \tilde{b} \in \mathfrak{B}\}$ is self-stable w.r.t. $\tilde{\rho}'$.

The next lemma follows from Corollary 3.3 of [DOV].

Lemma 1.6. *Let (X, d) be a metric space and let Y, Z be dense subsets of X . Then for every $p \in Y \cap Z$ and every $\Omega_{p, \tilde{r}}^Y$ there are $\Omega_{p, \tilde{r}}^Z$ and an isometry $f : \Omega_{p, \tilde{r}}^Y \rightarrow \Omega_{p, \tilde{r}}^Z$ such that $f(\alpha_Y) = \alpha_Z$ where α_Y and α_Z are the marked points of $\Omega_{p, \tilde{r}}^Y$ and $\Omega_{p, \tilde{r}}^Z$ respectively.*

2. Completely strongly porous sets

The notion of “porosity” for the first time appeared in some early works of Denjoy [D1], [D2] and Khintchine [CHI] and then arose independently in the study of cluster sets in 1967 by Dolženko [DOL]. A useful collection of facts related to the notion of porosity can be found, for example, in [FH], [HV], [T] and [TK]. The porosity appears naturally in many problems and plays an implicit role in various regions of analysis (e. g., the cluster sets [Z1], the Julia sets [PR], the quasimetric maps [VA], the differential theory [KPS], the theory of generalized subharmonic functions [DR] and so on). The reader can also consult [Z2] and [Z3] for more information.

Let us recall the definition of the right upper porosity. Let E be a subset of $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. [T] The right upper porosity of E at 0 is the quantity

$$(2.1) \quad p^+(E, 0) := \limsup_{h \rightarrow 0^+} \frac{\lambda(E, 0, h)}{h}$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$, which could be the empty set \emptyset , that contains no points of E . The set E is strongly porous at 0 if $p^+(E, 0) = 1$.

Let $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. We shall say that $\tilde{\tau}$ is eventually decreasing if the inequality $\tau_{n+1} \leq \tau_n$ holds for sufficiently large n . Write \tilde{E}_0^d for the set of eventually decreasing sequences $\tilde{\tau}$ with $\lim_{n \rightarrow \infty} \tau_n = 0$ and $\tau_n \in E \setminus \{0\}$ for $n \in \mathbb{N}$.

Define \tilde{I}_E^d to be the set of sequences of open intervals $(a_n, b_n) \subseteq \mathbb{R}^+, n \in \mathbb{N}$, meeting the conditions:

- Each (a_n, b_n) is a connected component of the set $\text{Ext} E = \text{Int}(\mathbb{R}^+ \setminus E)$, i.e., $(a_n, b_n) \cap E = \emptyset$ but

$$((a, b) \neq (a_n, b_n)) \Rightarrow ((a, b) \cap E \neq \emptyset)$$

for every $(a, b) \supseteq (a_n, b_n)$;

- $(a_n)_{n \in \mathbb{N}}$ is eventually decreasing;

- $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} \frac{b_n - a_n}{b_n} = 1$.

Define also an equivalence \asymp on the set of sequences of strictly positive numbers as follows. Let $\tilde{a} = (a_n)_{n \in \mathbb{N}}$ and $\tilde{\gamma} = (\gamma_n)_{n \in \mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are some constants $c_1, c_2 > 0$ such that $c_1 a_n < \gamma_n < c_2 a_n$ for $n \in \mathbb{N}$.

The next definition is an equivalent form of Definition 1.4 from [BD2].

Definition 2.2. Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$ -strongly porous at 0 if there is a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that $\tilde{\tau} \asymp \tilde{a}$ where $\tilde{a} = (a_n)_{n \in \mathbb{N}}$. The set E is completely strongly porous at 0 if E is $\tilde{\tau}$ -strongly porous at 0 for every $\tilde{\tau} \in \tilde{E}_0^d$.

We denote by **CSP** the set of all completely strongly porous at 0 subsets of \mathbb{R}^+ . It is clear that every $E \in \mathbf{CSP}$ is strongly porous at 0 but not conversely. Moreover, if 0 is an isolated point of $E \subseteq \mathbb{R}^+$, then $E \in \mathbf{CSP}$.

The next lemma immediately follows from Definition 2.2.

Lemma 2.3. Let $E \subseteq \mathbb{R}^+$, $\tilde{\gamma} \in \tilde{E}_0^d$, $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ and let $\tilde{a} = (a_n)_{n \in \mathbb{N}}$. The equivalence $\tilde{\gamma} \asymp \tilde{a}$ holds if and only if we have

$$\limsup_{n \rightarrow \infty} \frac{a_n}{\gamma_n} < \infty \quad \text{and} \quad \gamma_n \leq a_n$$

for sufficiently large n .

Define a set \mathbb{N}_{N_1} as $\{N_1, N_1 + 1, \dots\}$ for $N_1 \in \mathbb{N}$.

Definition 2.4. Let

$$\tilde{A} = \{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d \quad \text{and} \quad \tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d.$$

Write $\tilde{A} \preceq \tilde{L}$ if there are $N_1 \in \mathbb{N}$ and $f : \mathbb{N}_{N_1} \rightarrow \mathbb{N}$ such that $a_n = l_{f(n)}$ for every $n \in \mathbb{N}_{N_1}$. \tilde{L} is universal if $\tilde{B} \preceq \tilde{L}$ holds for every $\tilde{B} \in \tilde{I}_E^d$.

Let $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ be universal and let

$$(2.2) \quad M(\tilde{L}) := \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}}.$$

In what follows acE means the set of all accumulation points of a set E .

Theorem 2.5. [BD2] Let $E \subseteq \mathbb{R}^+$ be strongly porous at 0 and $0 \in acE$. Then $E \in \mathbf{CSP}$ if and only if there is an universal $\tilde{L} \in \tilde{I}_E^d$ such that $M(\tilde{L}) < \infty$.

Note that the quantity $M(\tilde{L})$ depends from the set E only (for details see [BD2]). The following lemma is used in next part of the paper.

Lemma 2.6. [BD2] *Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$. Then E is $\tilde{\tau}$ -strongly porous at 0 if and only if there is a constant $k \in (1, \infty)$ such that for every $K \in (k, \infty)$ there exists $N_1(K) \in \mathbb{N}$ with $(k\tau_n, K\tau_n) \cap E = \emptyset$ for every $n \geq N_1(K)$.*

Let us consider now a simple set belonging to **CSP**.

Example 2.7. Let $(x_n)_{n \in \mathbb{N}}$ be strictly decreasing sequence of positive real numbers with $\lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = 0$. Define a set W as

$$W = \{x_n : n \in \mathbb{N}\},$$

i.e., W is the range of the sequence $(x_n)_{n \in \mathbb{N}}$. Then $W \in \mathbf{CSP}$ and $\tilde{L} = \{(x_{n+1}, x_n)\}_{n \in \mathbb{N}} \in \tilde{I}_W^d$ is universal with $M(\tilde{L}) = 1$.

Proposition 2.8. *Let $E \subseteq \mathbb{R}^+$. Then the inclusion*

$$(2.3) \quad \{E \cup A : A \in \mathbf{CSP}\} \subseteq \mathbf{CSP}$$

holds if and only if 0 is an isolated point of E .

Remark 2.9. Inclusion (2.3) means that $E \cup A \in \mathbf{CSP}$ for every $A \in \mathbf{CSP}$.

Proof of Proposition 2.8. If $0 \notin acE$, then (2.3) follows almost directly and we omit the details here. Suppose $0 \in acE$. Then there is a sequence $(\tau_n)_{n \in \mathbb{N}}$ such that $\tau_n \in E$ and $\tau_{n+1} \leq 2^{-n^2} \tau_n$ for every $n \in \mathbb{N}$. Let $M_1, M_2, \dots, M_k, \dots$ be an infinite partition of \mathbb{N} ,

$$\bigcup_{k=1}^{\infty} M_k = \mathbb{N}, \quad M_i \cap M_j = \emptyset \text{ if } i \neq j$$

such that $\text{card} M_k = \text{card} \mathbb{N}$ for every k and $\nu(1) < \nu(2) < \dots < \nu(k) \dots$ where

$$(2.4) \quad \nu(k) = \min_{n \in M_k} n.$$

Let $n \in \mathbb{N}$ and let $m(n)$ be the index such that $n \in M_{m(n)}$. For every $n \in \mathbb{N}$ define τ_n^* as $2^{-m(n)} \tau_n$. Write

$$E_1 = \{\tau_n : n \in \mathbb{N}\} \quad \text{and} \quad E_1^* = \{\tau_n^* : n \in \mathbb{N}\}.$$

Using Lemma 2.6 we can show that $E_1 \cup E_1^*$ is not τ^* -strongly porous with $\tilde{\tau}^* = (\tau_n^*)_{n \in \mathbb{N}}$. Consequently $E_1 \cup E_1^* \notin \mathbf{CSP}$. It implies that $E \cup E_1^* \notin \mathbf{CSP}$ because $E_1 \subseteq E$. To complete the proof, it suffices to show that $E_1^* \in \mathbf{CSP}$. To this end, we note that (2.4) and the inequalities $\nu(1) < \nu(2) < \dots < \nu(k) \dots$ imply that $m(n) \leq n$ for every $n \in \mathbb{N}$. Indeed, if $m(n) = k$, then we have

$$\begin{aligned} n \geq \nu(k) &= (\nu(k) - \nu(k-1)) + (\nu(k-1) - \nu(k-2)) + \dots + (\nu(2) - \nu(1)) + \nu(1) \\ &\geq (k-1) + \nu(1) = k = m(n). \end{aligned}$$

Consequently

$$\tau_n^* = 2^{-m(n)}\tau_n \geq 2^{-n}\tau_n \geq 2^{-n^2}\tau_n \geq \tau_{n+1} \geq \tau_{n+1}^*$$

for every $n \in \mathbb{N}$. It follows from that

$$\lim_{n \rightarrow \infty} \frac{\tau_n^*}{\tau_{n+1}^*} \geq \lim_{n \rightarrow \infty} \frac{2^{-n}\tau_n}{2^{-n^2}\tau_n} = \lim_{n \rightarrow \infty} 2^{n^2-n} = +\infty.$$

Thus, as in Example 2.7, we have $E_1^* \in \mathbf{CSP}$. ■

3. Uniform boundedness and \mathbf{CSP}

Let $\mathfrak{F} = \{(X_i, d_i) : i \in I\}$ be a nonempty family of metric spaces. The family \mathfrak{F} is *uniformly bounded* if there is a constant $c > 0$ such that the inequality $\text{diam} X_i < c$ holds for every $i \in I$. If all metric spaces $(X_i, d_i) \in \mathfrak{F}$ are pointed with marked points $p_i \in X_i$, then the uniform boundedness of \mathfrak{F} can be described by the next way. Define

$$(3.1) \quad \rho^*(X_i) := \sup_{x \in X_i} d_i(x, p_i) \quad \text{and} \quad R^*(\mathfrak{F}) := \sup_{i \in I} \rho^*(X_i).$$

The family \mathfrak{F} is uniformly bounded if and only if $R^*(\mathfrak{F}) < \infty$.

Proposition 3.1. *Let (X, d, p) be a pointed metric space and let $\Omega_{\mathbf{p}}^{\mathbf{X}}$ be the set of all pretangent spaces to X at p . The following statements are equivalent.*

- (i₁) *The family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ is uniformly bounded.*
- (i₂) *The point p is an isolated point of X .*

Proof. The implication (i₂) \Rightarrow (i₁) follows directly from the definitions. To prove (i₁) \Rightarrow (i₂) suppose that $p \in \text{ac} X$. Let $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{X}$ be a sequence of distinct points of X such that $\lim_{n \rightarrow \infty} d(x_n, p) = 0$. For $t > 0$ define the scaling sequence $\tilde{r}_t = (r_{n,t})_{n \in \mathbb{N}}$ with $r_{n,t} = \frac{d(x_n, p)}{t}$. It follows at once from Definition 1.1 that \tilde{x} and \tilde{p} are mutually stable w.r.t \tilde{r}_t and

$$(3.2) \quad \tilde{d}_{\tilde{r}_t}(\tilde{x}, \tilde{p}) = t.$$

Let $\tilde{X}_{p, \tilde{r}_t}$ be a maximal self-stable family meeting the relation $\tilde{x} \in \tilde{X}_{p, \tilde{r}_t}$. Equality (3.2) implies the inequality

$$\text{diam } \Omega_{p, \tilde{r}_t}^X \geq t,$$

where $\Omega_{p, \tilde{r}_t}^X = \pi(\tilde{X}_{p, \tilde{r}_t})$. Consequently the family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ is not uniformly bounded. The implication (i₁) \Rightarrow (i₂) follows. ■

The proposition above shows that the question on the uniform boundedness can be informative only for some special subfamilies of $\Omega_{\mathbf{p}}^{\mathbf{X}}$. We can narrow down the family $\Omega_{\mathbf{p}}^{\mathbf{X}}$ by the way of consideration some special scaling sequences.

Definition 3.2. Let (X, d, p) be a pointed metric space and let $p \in acX$. A scaling sequence $(r_n)_{n \in \mathbb{N}}$ is normal if $(r_n)_{n \in \mathbb{N}}$ is eventually decreasing and there is $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that

$$(3.3) \quad \lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n} = 1.$$

Proposition 3.3. *The following statements hold for every pointed metric space (X, d, p) .*

- (i₁) *If $\Omega_{p, \tilde{r}}^X$ contains at least two distinct points, then there are $c > 0$ and a subsequence $(r_{n_k})_{k \in \mathbb{N}}$ of \tilde{r} so that the sequence $(cr_{n_k})_{k \in \mathbb{N}}$ is normal.*
- (i₂) *If $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ and (3.3) holds, then there is an infinite increasing sequence $(n_k)_{k \in \mathbb{N}}$ so that $(d(x_{n_k}, p))_{k \in \mathbb{N}}$ is decreasing.*
- (i₃) *If \tilde{r} is a normal scaling sequence, then there is $(x_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that (3.3) holds and $(d(x_n, p))_{n \in \mathbb{N}}$ is eventually decreasing.*

Proof. It is easily verified that (i₁) and (i₂) hold. To verify (i₃) observe that there is $(y_n)_{n \in \mathbb{N}} \in \tilde{X}$ which satisfies $d(y_n, p) > 0$ for every $n \in \mathbb{N}$ and (3.3) with $(x_n)_{n \in \mathbb{N}} = (y_n)_{n \in \mathbb{N}}$. Let $m(n) \in \mathbb{N}$ meet the conditions $m(n) \leq n$ and $d(y_{m(n)}, p) = \min_{1 \leq i \leq n} d(y_i, p)$. The conditions $\lim_{n \rightarrow \infty} y_n = p$ and $d(y_n, p) > 0$ for $n \in \mathbb{N}$ imply that $m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Since \tilde{r} is eventually decreasing, there is $n_0 \in \mathbb{N}$ such that $(r_{m(n)}) = r_n$ for every $n \geq n_0$. Consequently, we obtain

$$1 = \lim_{n \rightarrow \infty} \frac{d(y_{m(n)}, p)}{r_{m(n)}} = \lim_{n \rightarrow \infty} \frac{d(y_{m(n)}, p)}{r_n} \leq \lim_{n \rightarrow \infty} \frac{d(y_n, p)}{r_n} = 1.$$

It is clear that $(d(y_{m(n)}, p))_{n \in \mathbb{N}}$ is decreasing. Thus (i₃) holds with $(x_n)_{n \in \mathbb{N}} = (y_{m(n)})_{n \in \mathbb{N}}$. ■

Write $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ for the set of pretangent spaces $\Omega_{p, \tilde{r}}^X$ with normal scaling sequences. Under what conditions the family $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded?

Remark 3.4. Of course, the property of scaling sequence \tilde{r} to be normal depends on the underlying space (X, d, p) . Nevertheless for every pointed metric space (X, d, p) a scaling sequence \tilde{r} is normal for this space if and only if it is normal for the space $(S_p(X), |\cdot|, 0)$. We shall use this simple fact below in Proposition 3.5.

In the next proposition we define $\Omega_0^{\mathbf{E}}(\mathbf{n})$ to be the set of all pretangent spaces to the distance set $E = S_p(X)$ at 0 w.r.t. normal scaling sequences.

Proposition 3.5. *Let (X, d, p) be a pointed metric space and let $E = S_p(X)$. Then we have*

$$(3.4) \quad R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) = R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n}))$$

where $R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n}))$ and $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n}))$ are defined by (3.1) with $\mathfrak{F} = \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ and $\mathfrak{F} = \Omega_0^{\mathbf{E}}(\mathbf{n})$ respectively.

Proof. If $p \notin acX$, then the set of normal scaling sequences is empty. Consequently we have $\Omega_0^{\mathbf{E}}(\mathbf{n}) = \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n}) = \emptyset$, so we suppose that $p \in acX$.

For each normal scaling sequence \tilde{r} and every $\tilde{x} \in \tilde{X}$ having the finite limit $\lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n}$ we can find $(s_n)_{n \in \mathbb{N}} \in \tilde{E}$ such that

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{d(x_n, p)}{r_n} = \lim_{n \rightarrow \infty} \frac{s_n}{r_n}.$$

Hence the inequality

$$(3.6) \quad \rho(\alpha, \beta) \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n}))$$

holds with $\alpha = \pi(\tilde{p})$ for every $\beta \in \Omega_{p, \tilde{r}}^X$ and every $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$. Taking supremum over all $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ and $\beta \in \Omega_{p, \tilde{r}}^X$, we get

$$(3.7) \quad R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \geq R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})).$$

It still remains to prove the inequality

$$(3.8) \quad R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \leq R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})).$$

As is easily seen, for every normal scaling \tilde{r} and every $\tilde{s} \in \tilde{E}$ with $\lim_{n \rightarrow \infty} \frac{s_n}{r_n} < \infty$, there is $\tilde{x} \in \tilde{X}$ satisfying (3.5). Now reasoning as in the proof of (3.7) we obtain (3.8). Equality (3.4) follows from (3.7) and (3.8). \blacksquare

Lemma 3.6. *Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. If the inequality*

$$(3.9) \quad R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) < \infty$$

holds, then $E \in \mathbf{CSP}$.

Proof. Suppose that (3.9) holds but there is $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$ such that E is not $\tilde{\tau}$ -strongly porous at 0. Then, by Lemma 2.6, for every $k > 1$ there is $K \in (k, \infty)$ such that $(k\tau_n, K\tau_n) \cap E \neq \emptyset$ for all n belonging to an infinite set $A \subseteq \mathbb{N}$. Let us put

$$(3.10) \quad k = 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

It simply follows from (3.1) and Definition 3.2 that $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \geq 1$. Thus $k \geq 2$. Consequently we can find $K \in (k, \infty)$ and an infinite set $A = \{n_1, \dots, n_j, \dots\} \subseteq \mathbb{N}$, such that for every $n_j \in A$ there is $x_j \in E$ satisfying the double inequality

$$(3.11) \quad k < \frac{x_j}{\tau_{n_j}} < K.$$

Thus the sequence $\left(\frac{x_j}{\tau_{n_j}}\right)_{j \in \mathbb{N}}$ is bounded. Hence it contains a convergent subsequence. Passing to this subsequence we obtain

$$(3.12) \quad \lim_{j \rightarrow \infty} \frac{x_j}{\tau_{n_j}} < \infty.$$

Now (3.10) and (3.11) imply

$$(3.13) \quad \lim_{j \rightarrow \infty} \frac{x_j}{\tau_{n_j}} = \lim_{j \rightarrow \infty} \frac{|0 - x_j|}{\tau_{n_j}} \geq 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

The scaling sequence $\tilde{r} = (r_j)_{j \in \mathbb{N}}$ with $r_j = \tau_{n_j}$, $j \in \mathbb{N}$, is normal. The existence of finite limit (3.12) implies that $\tilde{x} = (x_j)_{j \in \mathbb{N}}$ and $\tilde{0}$ are mutually stable w.r.t \tilde{r} . Consequently there is a maximal self-stable family $\tilde{E}_{0, \tilde{r}}$ such that $\tilde{x}, \tilde{0} \in \tilde{E}_{0, \tilde{r}}$. Write $\Omega_{0, \tilde{r}}^E$ for the metric identification of $\tilde{E}_{0, \tilde{r}}$ and α for the natural projection of $\tilde{0}$. Using (3.13) and (3.1), we obtain

$$R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) \geq \sup_{\gamma \in \Omega_{0, \tilde{r}}^E} \rho(\alpha, \gamma) \geq 2R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

The last double inequality is inconsistent because $1 \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) < \infty$. Thus if (3.9) holds, then E is \tilde{r} -strongly porous at 0, as required. \blacksquare

Let $\tilde{\tau} \in \tilde{E}_0^d$. Define a subset $\tilde{I}_E^d(\tilde{\tau})$ of the set \tilde{I}_E^d by the rule:

$$\begin{aligned} (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})) &\Leftrightarrow (\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d \text{ and} \\ &\tau_n \leq a_n \text{ for sufficiently large } n \in \mathbb{N}). \end{aligned}$$

Write

$$(3.14) \quad C(\tilde{\tau}) := \inf(\limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n}) \quad \text{and} \quad C_E := \sup_{\tilde{\tau} \in \tilde{E}_0^d} C(\tilde{\tau})$$

where the infimum is taken over all $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})$.

Using Theorem 2.5 we can prove the following

Proposition 3.7. [BD2] *Let $E \subseteq \mathbb{R}^+$ and $\tilde{\tau} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$ -strongly porous at 0 if and only if $C(\tilde{\tau}) < \infty$. The membership $E \in \mathbf{CSP}$ holds if and only if $C_E < \infty$.*

Remark 3.8. If $E \subseteq \mathbb{R}^+$, $i = 1, 2$, $\{(a_n^{(i)}, b_n^{(i)})\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ and $\tilde{a}^1 \asymp \tilde{a}^2$ where $\tilde{a}^i = (a_n^{(i)})_{n \in \mathbb{N}}$, then there is $n_0 \in \mathbb{N}$ such that $(a_n^{(1)}, b_n^{(1)}) = (a_n^{(2)}, b_n^{(2)})$ for every $n \geq n_0$. Consequently if E is $\tilde{\tau}$ -strongly porous and $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d(\tilde{\tau})$, then we have

$$\text{either } \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} = \infty \quad \text{or} \quad \limsup_{n \rightarrow \infty} \frac{a_n}{\tau_n} = C(\tilde{\tau}) < \infty.$$

Lemma 3.9. [BD2] *Let $E \in \mathbf{CSP}$. If $\tilde{L} \in \tilde{I}_E^d$ is universal, then $M(\tilde{L}) = C_E$ where the quantities $M(\tilde{L})$ and C_E are defined by (2.2) and (3.14) respectively.*

Proposition 3.10. *Let $E \subseteq \mathbb{R}^+$ and let $0 \in acE$. Then the equality*

$$(3.15) \quad C_E = R^*(\Omega_0^{\mathbf{E}}(\mathbf{n}))$$

holds.

Proof. Let us prove the inequality

$$(3.16) \quad C_E \geq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

This is trivial if $C_E = \infty$. Suppose that $C_E < \infty$. Inequality (3.16) holds if, for every normal scaling sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$ and each $\tilde{y} = (y_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$, the existence of the finite limit $\lim_{n \rightarrow \infty} \frac{y_n}{r_n}$ implies the inequality

$$(3.17) \quad \lim_{n \rightarrow \infty} \frac{y_n}{r_n} \leq C_E.$$

Since \tilde{r} is normal, Proposition 3.3 implies that there is $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{E}_0^d$ with $\lim_{n \rightarrow \infty} \frac{r_n}{x_n} = 1$. Consequently (3.17) holds if and only if

$$(3.18) \quad \lim_{n \rightarrow \infty} \frac{y_n}{x_n} \leq C_E.$$

If $\lim_{n \rightarrow \infty} \frac{y_n}{x_n} = 0$, then (3.18) is trivial. Suppose that $0 < \lim_{n \rightarrow \infty} \frac{y_n}{x_n} < \infty$. The last double inequality implies the equivalence $\tilde{x} \asymp \tilde{y}$. In accordance with Proposition 3.7, $E \in \mathbf{CSP}$ if and only if $C_E < \infty$ holds. Hence E is \tilde{x} -strongly porous at 0. Consequently there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that $\tilde{x} \asymp \tilde{a}$. The relations $\tilde{x} \asymp \tilde{y}$ and $\tilde{x} \asymp \tilde{a}$ imply $\tilde{y} \asymp \tilde{a}$. Using Lemma 2.3 we can find $N_0 \in \mathbb{N}$ such that $y_n \leq a_n$ for $n \geq N_0$. Consequently we have $\frac{y_n}{x_n} \leq \frac{a_n}{x_n}$ for $n \geq N_0$, which implies

$$\lim_{n \rightarrow \infty} \frac{y_n}{x_n} \leq \limsup_{n \rightarrow \infty} \frac{a_n}{x_n} \leq C(\tilde{\tau}) \leq C_E$$

(see (3.14)). Inequality (3.16) follows.

To prove (3.15), it still remains to verify the inequality

$$(3.19) \quad C_E \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

It is trivial if $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) = \infty$. Suppose that

$$(3.20) \quad R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) < \infty.$$

Inequality (3.19) holds if

$$(3.21) \quad C(\tilde{x}) \leq R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})).$$

for every $\tilde{x} \in \tilde{E}_0^d$. Let E^1 denote the closure of the set E in \mathbb{R}^+ and let $\tilde{x} \in \tilde{E}_0^d$. It follows at once from Lemma 1.6 that $R^*(\Omega_0^{\mathbf{E}}(\mathbf{n})) = R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n}))$. Consequently

(3.21) holds if $C(\tilde{x}) \leq R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n}))$. By Lemma 3.6, inequality (3.20) implies that $E \in \mathbf{CSP}$. Hence, by Lemma 2.3, there is $\{(a_n, b_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ such that

$$(3.22) \quad \limsup_{n \rightarrow \infty} \frac{a_n}{x_n} < \infty$$

and $a_n \geq x_n$ for sufficiently large n . Inequality (3.22) implies the equality

$$(3.23) \quad C(\tilde{x}) = \limsup_{n \rightarrow \infty} \frac{a_n}{x_n},$$

(see Remark 3.8). Let $(n_j)_{j \in \mathbb{N}}$ be an infinite increasing sequence for which

$$(3.24) \quad \lim_{j \rightarrow \infty} \frac{a_{n_j}}{x_{n_j}} = \limsup_{n \rightarrow \infty} \frac{a_n}{x_n}.$$

Define $r_j := x_{n_j}$, $\tilde{r} := (r_j)_{j \in \mathbb{N}}$ and $t_j := a_{n_j}$, $\tilde{t} := (t_j)_{j \in \mathbb{N}}$. It is clear that \tilde{r} is a normal scaling sequence. Relation (3.22) and (3.24) imply that \tilde{t} and $\tilde{0} = (0, 0, \dots, 0, \dots)$ are mutually stable w.r.t. \tilde{r} . Let $\tilde{E}_{0, \tilde{r}}^1$ be a maximal (in \tilde{E}^1) self-stable family containing \tilde{t} and $\tilde{0}$. Using (3.1), (3.23) and (3.24), we obtain

$$R^*(\Omega_0^{\mathbf{E}^1}(\mathbf{n})) \geq \sup_{\tilde{y} \in \tilde{E}_{0, \tilde{r}}^1} \tilde{d}_{\tilde{r}}(\tilde{y}, \tilde{0}) \geq \tilde{d}_{\tilde{r}}(\tilde{t}, \tilde{0}) = C(\tilde{x}).$$

Hence (3.19) holds that completes the proof of (3.15). ■

The following theorem gives the necessary and sufficient conditions under which $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded.

Theorem 3.11. *Let (X, d, p) be a pointed metric space and let $E = S_p(X)$. The family $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded if and only if $E \in \mathbf{CSP}$. If $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded and $p \in acX$, then*

$$(3.25) \quad R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})) = M(\tilde{L})$$

where \tilde{L} is an universal element of (\tilde{I}_E^d, \preceq) and $M(\tilde{L})$ is defined by (2.2).

Proof. The theorem is trivial if p is an isolated point of X , so that we assume $p \in acX$. By Proposition 3.5, $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded if and only if $\Omega_0^{\mathbf{E}}(\mathbf{n})$ is uniformly bounded. Since $C_E = R^*(\Omega_0^{\mathbf{E}}(\mathbf{n}))$ (see (3.15)), $\Omega_0^{\mathbf{E}}(\mathbf{n})$ is uniformly bounded if and only if $C_E < \infty$. Using Proposition 3.7 we obtain that $\Omega_0^{\mathbf{E}}(\mathbf{n})$ is uniformly bounded if and only if $E \in \mathbf{CSP}$.

Let us prove that (3.25) holds if $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded and $p \in acX$. In this case, as was proved above, $E \in \mathbf{CSP}$. Consequently, by Theorem 2.5, there is an universal element $\tilde{L} \in \tilde{I}_E^d$ such that $M(\tilde{L}) < \infty$. Lemma 3.9 implies that

$$(3.26) \quad M(\tilde{L}) = C_E.$$

By Proposition 3.10, we also have the equality

$$(3.27) \quad C_E = R^*(\Omega_0^E(\mathbf{n})).$$

Since $R^*(\Omega_0^E(\mathbf{n})) = R^*(\Omega_{\mathbf{p}}^X(\mathbf{n}))$, equalities (3.26) and (3.27) imply (3.25). \blacksquare

Remark 3.12. It is known (see [DAK]) that a bounded tangent space to X at p exists if and only if $S_p(X) = \{d(x, p) : x \in X\}$ is strongly porous at 0. The necessary and sufficient conditions under which all pretangent spaces to X at p are bounded also formulated in terms of the local porosity of the set $S_p(X)$ (see [BD1] for details).

Theorem 3.11 is an example of translation of some results related to completely strongly porous at 0 sets on the language of pretangent spaces. For more examples of such translation see Theorem 4.6 and Proposition 5.4 of the paper.

Proposition 3.13. *Let (X, d, p) be a pointed metric space and let $\Omega_{\mathbf{p}}^X(\mathbf{n}) \neq \emptyset$. If the family $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly bounded, then there is $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^X(\mathbf{n})$ such that the equality*

$$(3.28) \quad R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = \rho^*(\Omega_{p, \tilde{r}}^X)$$

holds.

Proof. Suppose that $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly bounded. Write $E := S_p(X)$. In the correspondence with Theorem 3.11, we have the equality

$$(3.29) \quad R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = M(\tilde{L}),$$

where $\tilde{L} = \{(l_n, m_n)\}_{n \in \mathbb{N}} \in \tilde{I}_E^d$ is an universal element of (\tilde{I}_E^d, \preceq) and

$$M(\tilde{L}) = \limsup_{n \rightarrow \infty} \frac{l_n}{m_{n+1}} < \infty.$$

Let us consider a subsequence $\{(l_{n_k}, m_{n_k})\}_{k \in \mathbb{N}}$ of the sequence \tilde{L} such that

$$(3.30) \quad \lim_{k \rightarrow \infty} \frac{l_{n(k)}}{m_{n(k)+1}} = M(\tilde{L}).$$

Since each (l_n, m_n) is a connected component of $ExtE = Int(\mathbb{R}^+ \setminus E)$, there is a sequence $(t_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $t_k \in E$ and

$$(3.31) \quad \lim_{k \rightarrow \infty} \frac{t_k}{l_{n(k)}} = 1.$$

Limit relations (3.30) and (3.31) imply

$$(3.32) \quad \lim_{k \rightarrow \infty} \frac{t_k}{m_{n(k)+1}} = M(\tilde{L}).$$

Passing, if it is necessary, to a subsequence again we may suppose that there is a decreasing sequence $(s_k)_{k \in \mathbb{N}}$ such that

$$(3.33) \quad \lim_{k \rightarrow \infty} \frac{s_k}{m_{n(k)+1}} = 1$$

and $s_k \in E$ for every $k \in \mathbb{N}$. Write $r_k := m_{n(k)+1}$ for every $k \in \mathbb{N}$. Let $(y_k)_{k \in \mathbb{N}} \in \tilde{X}$ be a sequence such that $t_k = d(p, y_k)$, $k \in \mathbb{N}$. The triangle inequality implies that

$$d(x_k, y_k) \leq d(x_k, p) + d(p, y_k).$$

The last inequality, (3.32) and (3.33) imply that

$$\limsup_{k \rightarrow \infty} \frac{d(x_k, y_k)}{r_k} \leq \lim_{k \rightarrow \infty} \frac{d(x_k, p)}{r_k} + \lim_{k \rightarrow \infty} \frac{d(y_k, p)}{r_k} = 1 + M(\tilde{L}) < \infty.$$

Hence, the sequence $\left(\frac{d(x_k, y_k)}{r_k}\right)_{k \in \mathbb{N}}$ contains a convergent subsequence. Without loss of generality we may suppose that there is a finite limit

$$\lim_{k \rightarrow \infty} \frac{d(x_k, y_k)}{r_k}.$$

Thus, the family $\{\tilde{x}, \tilde{y}, \tilde{p}\}$, where $\tilde{x} = (x_k)_{k \in \mathbb{N}}$, $\tilde{y} = (y_k)_{k \in \mathbb{N}}$ and $\tilde{p} = (p, p, \dots)$ is self-stable. By the Zorn lemma, there is a maximal self-stable family $\tilde{X}_{p, \tilde{r}}$ such that

$$(3.34) \quad \{\tilde{x}, \tilde{y}, \tilde{p}\} \subseteq \tilde{X}_{p, \tilde{r}}.$$

Let $\Omega_{p, \tilde{r}}^X$ be the pretangent space obtained by metric identification of $\tilde{X}_{p, \tilde{r}}$. Since $(s_k)_{k \in \mathbb{N}}$ is decreasing and equation (3.33) holds, the scaling sequence $(r_k)_{k \in \mathbb{N}}$ is normal, i.e. $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$. Let $\alpha = \pi(\tilde{p})$, $\tilde{p} = (p, p, \dots)$, and $\beta = \pi(\tilde{y})$, where π is the natural projection of $\tilde{X}_{p, \tilde{r}}$ on $\Omega_{p, \tilde{r}}^X$. It follows from (3.32), that

$$\rho^*(\Omega_{p, \tilde{r}}^X) = \sup_{\gamma \in \Omega_{p, \tilde{r}}^X} \rho(\alpha, \gamma) \geq \rho(\alpha, \beta) = M(\tilde{L}).$$

The inequality $\rho^*(\Omega_{p, \tilde{r}}^X) \geq M(\tilde{L})$ and (3.19) imply $R^*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})) \leq \rho^*(\Omega_{p, \tilde{r}}^X)$. The converse inequality is trivial. Equality (3.28) follows. \blacksquare

Remark 3.14. If $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly bounded and nonempty, then it can be proved that there is $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ for which the equality

$$\text{diam}(\Omega_{p, \tilde{r}}^X) = \sup_{\Omega \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})} \text{diam}(\Omega)$$

holds.

4. Uniform boundedness and uniform discreteness

Let I be an index set and let $\mathfrak{F} = \{(X_i, d_i, p_i) : i \in I\}$ be a nonempty family of pointed metric spaces. We set

$$(4.1) \quad \rho_*(X_i) := \begin{cases} \inf\{d_i(x, p_i) : x \in X_i \setminus \{p_i\}\} & \text{if } X_i \neq \{p_i\} \\ +\infty & \text{if } X_i = \{p_i\} \end{cases}$$

for $i \in I$ and write

$$R_*(\mathfrak{F}) := \inf_{i \in I} \rho_*(X_i).$$

If $R_*(\mathfrak{F}) > 0$, then we say that \mathfrak{F} is *uniformly discrete* (w.r.t. the points p_i)

As in Proposition 3.1 it is easy to show that the family Ω_p^X of all pretangent spaces is uniformly discrete if and only if p is an isolated point of the metric space X . Thus, it make sense to consider $\Omega_p^X(\mathbf{n})$.

Theorem 4.1. *Let (X, d, p) be a pointed metric space such that $\Omega_p^X(\mathbf{n}) \neq \emptyset$. Then $\Omega_p^X(\mathbf{n})$ is uniformly discrete if and only if it is uniformly bounded. If $\Omega_p^X(\mathbf{n})$ is uniformly bounded, then the equality*

$$(4.2) \quad R_*(\Omega_p^X(\mathbf{n})) = \frac{1}{R^*(\Omega_p^X(\mathbf{n}))}$$

holds.

Remark 4.2. The condition $\Omega_p^X(\mathbf{n}) \neq \emptyset$ implies $R^*(\Omega_p^X(\mathbf{n})) > 0$.

We will prove Theorem 4.1 in some more general setting.

Let (X, d, p) be a pointed metric space and let $t > 0$. Write tX for the pointed metric space (X, td, p) , i.e. tX is the pointed metric space with the same underlying set X and the marked point p , but equipped with the new metric td instead of d .

Definition 4.3. Let \mathfrak{F} be a nonempty family of pointed metric spaces. \mathfrak{F} is weakly self-similar if for every $(Y, d, p) \in \mathfrak{F}$ and every nonzero $t \in S_p(Y)$ the space $\frac{1}{t}Y$ belongs to \mathfrak{F} .

Theorem 4.4. *Let $\mathfrak{F} = \{(X_i, d_i, p_i) : i \in I\}$ be a weakly self-similar family of pointed metric spaces. Suppose that the sphere*

$$S_i = \{x \in X_i : d_i(x, p_i) = 1\}$$

is nonvoid for every $i \in I$. Then \mathfrak{F} is uniformly bounded if and only if \mathfrak{F} is uniformly discrete w.r.t. the marked points $p_i, i \in I$. If \mathfrak{F} is uniformly bounded, then the equality

$$(4.3) \quad R_*(\mathfrak{F}) = \frac{1}{R^*(\mathfrak{F})}$$

holds.

Proof. Assume that \mathfrak{F} is uniformly bounded but not uniformly discrete. Then there is a sequence $(x_{i_k})_{k \in \mathbb{N}}$ such that

$$x_{i_k} \in X_{i_k}, x_{i_k} \neq p_{i_k} \text{ and } \lim_{k \rightarrow \infty} d_{i_k}(x_{i_k}, p_{i_k}) = 0$$

Since all S_i are nonvoid, we can find a sequence $(y_{i_k})_{k \in \mathbb{N}}$ for which $y_{i_k} \in X_{i_k}$ and $d_{i_k}(y_{i_k}, p_{i_k}) = 1$ for every $k \in \mathbb{N}$. Define t_k to be $d_{i_k}(x_{i_k}, p_{i_k})$, $k \in \mathbb{N}$. Since \mathfrak{F} is weakly self-similar, $t_k^{-1}X_{i_k} \in \mathfrak{F}$ holds for every $k \in \mathbb{N}$. Now we obtain

$$R^*(\mathfrak{F}) \geq \limsup_{k \rightarrow \infty} t_k^{-1} d_{i_k}(y_{i_k}, p_{i_k}) = \limsup_{k \rightarrow \infty} t_k^{-1} = \infty.$$

Hence \mathfrak{F} is not uniformly bounded, contrary to the assumption. Therefore if \mathfrak{F} is uniformly bounded, then \mathfrak{F} is uniformly discrete. Similarly we can prove that the uniform discreteness of \mathfrak{F} implies the uniform boundedness of this family.

Suppose now that $R^*(\mathfrak{F}) < \infty$. Let us prove equality (4.3). Define a quantity $Q(\mathfrak{F})$ by the rule

$$Q(\mathfrak{F}) = \sup_{i \in I} \frac{\rho^*(X_i)}{\rho_*(X_i)}$$

where $\rho_*(X_i)$ is defined by (4.1) and $\rho^*(X_i)$ by (3.1). The first part of the theorem implies that \mathfrak{F} is uniformly discrete. Hence $R_*(\mathfrak{F}) > 0$, that implies

$$\rho_*(X_i) > 0, i \in I.$$

Moreover, the inequality $R^*(\mathfrak{F}) < \infty$ gives us the condition $\rho_*(X_i) < \infty$. Thus $Q(\mathfrak{F})$ is correctly defined. We claim that the equality $Q(\mathfrak{F}) = R^*(\mathfrak{F})$ holds. Indeed let $(i_k)_{k \in \mathbb{N}}$ be a sequence of indexes $i_k \in I$ such that

$$(4.4) \quad \lim_{k \rightarrow \infty} \rho_*(X_{i_k}) = R^*(\mathfrak{F}).$$

Since all S_i are nonvoid, we have $\rho_*(X_{i_k}) \leq 1$ for every X_{i_k} . Consequently

$$(4.5) \quad Q(\mathfrak{F}) \geq \limsup_{k \rightarrow \infty} \frac{\rho^*(X_{i_k})}{\rho_*(X_{i_k})} \geq \limsup_{k \rightarrow \infty} \rho^*(X_{i_k}) = \lim_{k \rightarrow \infty} \rho^*(X_{i_k}) \geq R^*(\mathfrak{F}).$$

Let us consider a sequence $(i_m)_{m \in \mathbb{N}}$, $i_m \in I$, for which

$$(4.6) \quad Q(\mathfrak{F}) = \lim_{m \rightarrow \infty} \frac{\rho^*(X_{i_m})}{\rho_*(X_{i_m})}.$$

The quantity $\frac{\rho^*(X_i)}{\rho_*(X_i)}$ is invariant w.r.t. the passage from X_i to $\frac{1}{t}X_i$ if $t \in S_{p_i}(X_i)$. Consequently using the uniform discreteness of \mathfrak{F} and the inequality $\rho_*(X_i) \leq 1$ (which follows from the condition $S_i \neq \emptyset$, $i \in I$), we may assume that

$$(4.7) \quad \lim_{m \rightarrow \infty} \rho_*(X_{i_m}) = 1.$$

Limit relations (4.6) and (4.7) imply

$$Q(\mathfrak{F}) = \lim_{m \rightarrow \infty} \rho_*(X_{i_m}) \leq R^*(\mathfrak{F}).$$

The last inequality and (4.5) give us the equality $R^*(\mathfrak{F}) = Q(\mathfrak{F})$. Reasoning similarly we obtain the equality $Q(\mathfrak{F}) = \frac{1}{R_*(\mathfrak{F})}$. Equality (4.3) follows. ■

Let us define a subset ${}^1\Omega_{\mathbf{p}}^{\mathbf{X}}$ of the set $\Omega_{\mathbf{p}}^{\mathbf{X}}$ of all pretangent spaces to X at p by the rule:

$$((\Omega_{p,\tilde{r}}^X, \rho, \alpha) \in {}^1\Omega_{\mathbf{p}}^{\mathbf{X}}) \Leftrightarrow (\{\delta \in \Omega_{p,\tilde{r}}^X : \rho(\alpha, \delta) = 1\} \neq \emptyset)$$

where $\alpha = \pi(\tilde{p})$ is the marked point of the pretangent space $\Omega_{p,\tilde{r}}^X$ and ρ is the metric on $\Omega_{p,\tilde{r}}^X$. To apply Theorem 4.4 to the proof of Theorem 4.1 we need the following

Lemma 4.5. *Let (X, d, p) be a pointed metric space. If $\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$, then there is ${}^1\Omega_{p,\tilde{\mu}}^X \in {}^1\Omega_{\mathbf{p}}^{\mathbf{X}}$ such that*

$$(4.8) \quad \rho_*({}^1\Omega_{p,\tilde{\mu}}^X) \leq \rho_*(\Omega_{p,\tilde{r}}^X) \leq \rho^*(\Omega_{p,\tilde{r}}^X) \leq \rho^*({}^1\Omega_{p,\tilde{\mu}}^X).$$

Conversely, if ${}^1\Omega_{p,\tilde{r}}^X \in {}^1\Omega_{\mathbf{p}}^{\mathbf{X}}$, then there is $\Omega_{p,\tilde{\mu}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ such that

$$(4.9) \quad \rho_*(\Omega_{p,\tilde{\mu}}^X) \leq \rho_*({}^1\Omega_{p,\tilde{r}}^X) \leq \rho^*({}^1\Omega_{p,\tilde{r}}^X) \leq \rho^*(\Omega_{p,\tilde{\mu}}^X).$$

Proof. Let $\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ and let $\tilde{X}_{p,\tilde{r}}$ be the maximal self-stable family which metric identification coincides with $\Omega_{p,\tilde{r}}^X$. We can find some sequences $\tilde{a}^i = (a_n^i)_{n \in \mathbb{N}}$ and $\tilde{b}^i = (b_n^i)_{n \in \mathbb{N}}$, $i \in \mathbb{N}$ such that

$$(4.10) \quad \lim_{i \rightarrow \infty} \rho(\pi(\tilde{b}^i), \alpha) = \rho_*(\Omega_{p,\tilde{r}}^X) \quad \text{and} \quad \lim_{i \rightarrow \infty} \rho(\pi(\tilde{a}^i), \alpha) = \rho^*(\Omega_{p,\tilde{r}}^X).$$

Since $\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$, the scaling sequence \tilde{r} is normal. Consequently there is $\tilde{c} = (c_n)_{n \in \mathbb{N}} \in \tilde{X}$ such that

$$\lim_{n \rightarrow \infty} \frac{d(c_n, p)}{r_n} = 1.$$

Let us define a countable family $\mathfrak{B} \subseteq \tilde{X}$ as

$$\mathfrak{B} = \{\tilde{b}^i : i \in \mathbb{N}\} \cup \{\tilde{a}^i : i \in \mathbb{N}\} \cup \{\tilde{c}\}.$$

The family \mathfrak{B} satisfies the condition of Lemma 1.5. Consequently there is an infinite subsequence $\tilde{r}' = (r_{n_k})_{k \in \mathbb{N}}$ the scaling sequence \tilde{r} for which the family

$$\mathfrak{B}' = \{(b_{n_k}^i)_{k \in \mathbb{N}} : i \in \mathbb{N}\} \cup \{(a_{n_k}^i)_{k \in \mathbb{N}} : i \in \mathbb{N}\} \cup \{(c_{n_k})_{k \in \mathbb{N}}\}$$

is self-stable w.r.t. \tilde{r}' . Completing \mathfrak{B}' to a maximal self-stable family and passing to the metric identification, we obtain the desired pretangent space ${}^1\Omega_{p, \tilde{\mu}}^X$ with $\tilde{\mu} = \tilde{r}'$. The second statement of the lemma follows directly from statement (i_2) of Proposition 3.3. \blacksquare

Proof of Theorem 4.1. Lemma 4.5 implies the equalities

$$(4.11) \quad R^*({}^1\Omega_{\mathbf{p}}^X) = R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) \quad \text{and} \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = R_*({}^1\Omega_{\mathbf{p}}^X).$$

Now Theorem 4.1 follows from Theorem 4.4 and (4.11). \blacksquare

The sum of theorems 4.1 and 3.11 yields the following result.

Theorem 4.6. *Let (X, d, p) be a metric space with a marked point $p \in acX$. Then the following three conditions are equivalent.*

- (i) $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly bounded.
- (ii) $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly discrete.
- (iii) $S_p(X) \in \mathbf{CSP}$.

Moreover if $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly bounded, then

$$R^*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = M(\tilde{L}) \quad \text{and} \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = \frac{1}{M(\tilde{L})}$$

where the quantity $M(\tilde{L})$ was defined by (2.2).

The following proposition is an analog of Proposition 3.13.

Proposition 4.7. *Let (X, d, p) be a pointed metric space and let $\Omega_{\mathbf{p}}^X(\mathbf{n}) \neq \emptyset$. If the family $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly discrete, then there is $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^X(\mathbf{n})$ such that the equality*

$$R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = \rho_*(\Omega_{p, \tilde{r}}^X)$$

holds.

A proof is completely similar to the proof of Proposition 3.13.

5. Metric space valued derivatives on uniformly bounded pretangent spaces

Let (X_i, d_i, p_i) be pointed metric spaces and let $\tilde{r}_i = \{r_n^{(i)}\}_{n \in \mathbb{N}}$ be scaling sequences, $i = 1, 2$. For a function $f : X_1 \rightarrow X_2$ define the mapping $\tilde{f} : \tilde{X}_1 \rightarrow \tilde{X}_2$ as

$$\tilde{f}(\tilde{x}) = (f(x_1), f(x_2), \dots, f(x_n), f(x_{n+1}), \dots)$$

if $\tilde{x} = (x_i)_{i \in \mathbb{N}} \in \tilde{X}_1$. Let us consider two maximal self-stable families $\tilde{X}_{p_1, \tilde{r}_1}^1 \subseteq \tilde{X}_1$ and $\tilde{X}_{p_2, \tilde{r}_2}^2 \subseteq \tilde{X}_2$.

Definition 5.1. A function $f : X_1 \rightarrow X_2$ is differentiable w.r.t. the pair $(\tilde{X}_{p_1, \tilde{r}_1}^1, \tilde{X}_{p_2, \tilde{r}_2}^2)$ if the following conditions are satisfied:

- (i) $\tilde{f}(\tilde{x}) \in \tilde{X}_{p_2, \tilde{r}_2}^2$ for every $\tilde{x} \in \tilde{X}_{p_1, \tilde{r}_1}^1$;
- (ii) The implication $(\tilde{d}_{\tilde{r}_1}(\tilde{x}, \tilde{y}) = 0) \Rightarrow (\tilde{d}_{\tilde{r}_2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) = 0)$ is true for all $\tilde{x}, \tilde{y} \in \tilde{X}_{p_1, \tilde{r}_1}^1$, where

$$\begin{aligned} \tilde{d}_{\tilde{r}_1}(\tilde{x}, \tilde{y}) &= \lim_{n \rightarrow \infty} \frac{d_1(x_n, y_n)}{r_n^{(1)}}, \\ \tilde{d}_{\tilde{r}_2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) &= \lim_{n \rightarrow \infty} \frac{d_2(f(x_n), f(y_n))}{r_n^{(2)}}. \end{aligned}$$

Remark 5.2. Note that condition (i) of Definition 5.1 implies the equality $f(p_1) = p_2$.

Let $\pi_{X_i} : \tilde{X}_{p_i, \tilde{r}_i}^i \rightarrow \Omega_{p_i, \tilde{r}_i}$, $i = 1, 2$, be natural projections. (See diagram (1.3)).

Definition 5.3. ([DOV]) A function $D^*f : \Omega_{p_1, \tilde{r}_1} \rightarrow \Omega_{p_2, \tilde{r}_2}$ is a metric space valued derivative of $f : X_1 \rightarrow X_2$ at the point $p_1 \in X_1$ if f is differentiable w.r.t. $(\tilde{X}_{p_1, \tilde{r}_1}^1, \tilde{X}_{p_2, \tilde{r}_2}^2)$ and the diagram

$$(5.1) \quad \begin{array}{ccc} \tilde{X}_{p_1, \tilde{r}_1}^1 & \xrightarrow{\tilde{f}} & \tilde{X}_{p_2, \tilde{r}_2}^2 \\ \pi_{X_1} \downarrow & & \downarrow \pi_{X_2} \\ \Omega_{p_1, \tilde{r}_1} & \xrightarrow{D^*f} & \Omega_{p_2, \tilde{r}_2} \end{array}$$

is commutative.

The existence of D^*f for differentiable w.r.t. $(\tilde{X}_{p_1, \tilde{r}_1}^1, \tilde{X}_{p_2, \tilde{r}_2}^2)$ functions f directly follows from Definition 5.1.

Let (X, d) be a metric space and Y be a set. Recall that a function $f : X \rightarrow Y$ is a locally constant at a point $p \in X$ if there is a neighborhood U of p such that f is constant on U . Write

$$(5.2) \quad c_p(f) := \begin{cases} \inf\{d(x, p) : f(x) \neq f(p)\} & \text{if } f \text{ is not constant on } X \\ \infty & \text{if } f \text{ is constant on } X. \end{cases}$$

Then f is locally constant at $p \in X$ if and only if $c_p(f) > 0$.

In the following proposition the quantity $M(\tilde{L})$ was defined by (2.2).

Proposition 5.4. *Let (X, d, p) be a pointed metric space with $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n}) \neq \emptyset$. Then the following statements are equivalent.*

- (i) $S_p(X) \in \mathbf{CSP}$.
- (ii) *There is a constant $c_X \in (0, \infty)$ such that if Y is a pointed metric space with a marked point a , $\tilde{X}_{p, \tilde{r}} \subseteq \tilde{X}$ and $\tilde{Y}_{a, \tilde{t}} \subseteq \tilde{Y}$ are maximal self-stable families, and $\Omega_{p, \tilde{r}}^X = \pi_X(\tilde{X}_{p, \tilde{r}}) \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ and $\Omega_{a, \tilde{t}}^Y = \pi_Y(\tilde{Y}_{a, \tilde{t}}) \in \Omega_{\mathbf{a}}^{\mathbf{Y}}$, and $f : X \rightarrow Y$ is differentiable w.r.t. $(\tilde{X}_{p, \tilde{r}}, \tilde{Y}_{a, \tilde{t}})$, then the derivative $D^*f : \Omega_{p, \tilde{r}}^X \rightarrow \Omega_{a, \tilde{t}}^Y$ satisfies the inequality*

$$(5.3) \quad c_\alpha(D^*f) \geq c_X,$$

where $\alpha = \pi_X(\tilde{p})$. Moreover, if condition (i) holds, then we can take $c_X = \frac{1}{M(\tilde{L})}$ and this bound is sharp for inequality (5.3).

Proof. Let (i) holds. Then, by Theorem 4.6, $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly discrete and

$$R_*(\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})) = \frac{1}{M(\tilde{L})} > 0.$$

Hence, for $r \in \left(0, \frac{1}{M(\tilde{L})}\right)$, the open ball $B_r(\alpha)$ of radius r centered at $\alpha = \pi_X(\tilde{p})$ contains the point α only. Consequently, for every function g defined on $\Omega_{p, \tilde{r}}^X \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ we have

$$c_\alpha(g) \geq \frac{1}{M(\tilde{L})}.$$

The implication (i) \Rightarrow (ii) follows. It is also shown that inequality (5.3) holds with $c_X = \frac{1}{M(\tilde{L})}$ for every $D^*(f) : \Omega_{p, \tilde{r}}^X \rightarrow \Omega_{a, \tilde{t}}^Y$ if statement (i) is valid. Suppose now that (ii) holds. By Theorem 4.6, statement (i) holds if and only if the family $\Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$ is uniformly discrete. Let $\tilde{X}_{p, \tilde{r}}$ be a maximal self-stable family and let $\Omega_{p, \tilde{r}}^X = \pi_X(\tilde{X}_{p, \tilde{r}}) \in \Omega_{\mathbf{p}}^{\mathbf{X}}(\mathbf{n})$. Let us consider the identical mappings

$$id_X : X \rightarrow X \quad \text{and} \quad id_\Omega : \Omega_{p, \tilde{r}}^X \rightarrow \Omega_{p, \tilde{r}}^X.$$

It is easy to show that id_X is differentiable w.r.t. $(\tilde{X}_{p,\tilde{r}}, \tilde{X}_{p,\tilde{r}})$ with the metric space value derivative $D^*(id_X) = id_\Omega$. By condition (ii), we have

$$c_\alpha(id_\Omega) \geq c_X.$$

Moreover, from the definitions of c_α and ρ_* it follows that

$$(5.4) \quad c_\alpha(id_\Omega) = \rho_*(\Omega_{p,\tilde{r}}^X) \quad \text{with} \quad \alpha = \pi_X(\tilde{p}).$$

Consequently, the inequality

$$(5.5) \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) \geq c_X$$

holds. Since $c_X > 0$, inequality (5.5) implies that $\Omega_{\mathbf{p}}^X(\mathbf{n})$ is uniformly discrete. To complete the proof, it suffices to show that $c_X = \frac{1}{M(\tilde{L})}$ is the best possible bound in (5.3). Indeed, by Proposition 4.7, there is ${}^1\Omega_{p,\tilde{r}}^X \in \Omega_{\mathbf{p}}^X(\mathbf{n})$ such that

$$(5.6) \quad \rho_*({}^1\Omega_{p,\tilde{r}}^X) = R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})).$$

From Theorem 4.6 it follows that

$$(5.7) \quad R_*(\Omega_{\mathbf{p}}^X(\mathbf{n})) = \frac{1}{M(\tilde{L})}.$$

Now using (5.4), (5.6) and (5.7), we obtain

$$c_\alpha(id_\Omega) = \frac{1}{M(\tilde{L})},$$

where id_Ω is the identical mapping of ${}^1\Omega_{p,\tilde{r}}^X$. The bound $c_X = \frac{1}{M(\tilde{L})}$ is sharp. ■

Remark 5.5. Recently, Dmytro Dordovskyi proved that pretangent spaces at regular points of k -dimensional parametric manifolds in \mathbb{R}^n are isometric to \mathbb{R}^k [DD]. Using this result, we can show that some metric space valued derivatives of mappings between such manifolds can be identified with linear mappings between Euclidean spaces. Every locally constant linear mapping is identically zero, so Proposition 5.4 gives us conditions under which every metrically differentiable function undergoes a “local vanishing” of metric derivative.

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